Legged Robots Locomotion: Footstep Planning and Realizing Motion Plans

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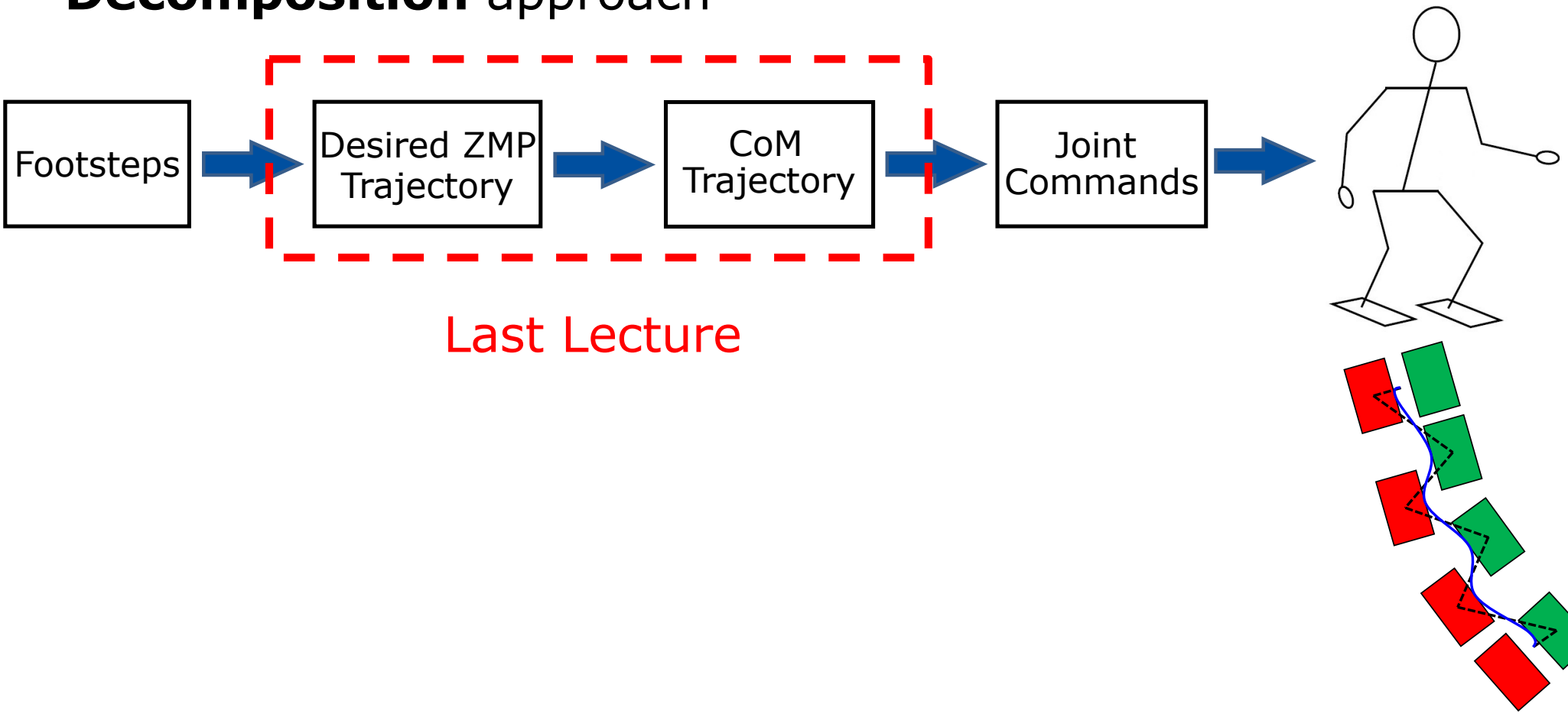
Goals of This Lecture

- Learn about **footstep planning** in a **2D grid map** representation
- Know about the fundamentals of **optimization**
- Understand **footstep planning** on **uneven terrain** with **convex optimization**
- Understand how planned motions are executed through **real-time** and **coordinated full-body control**

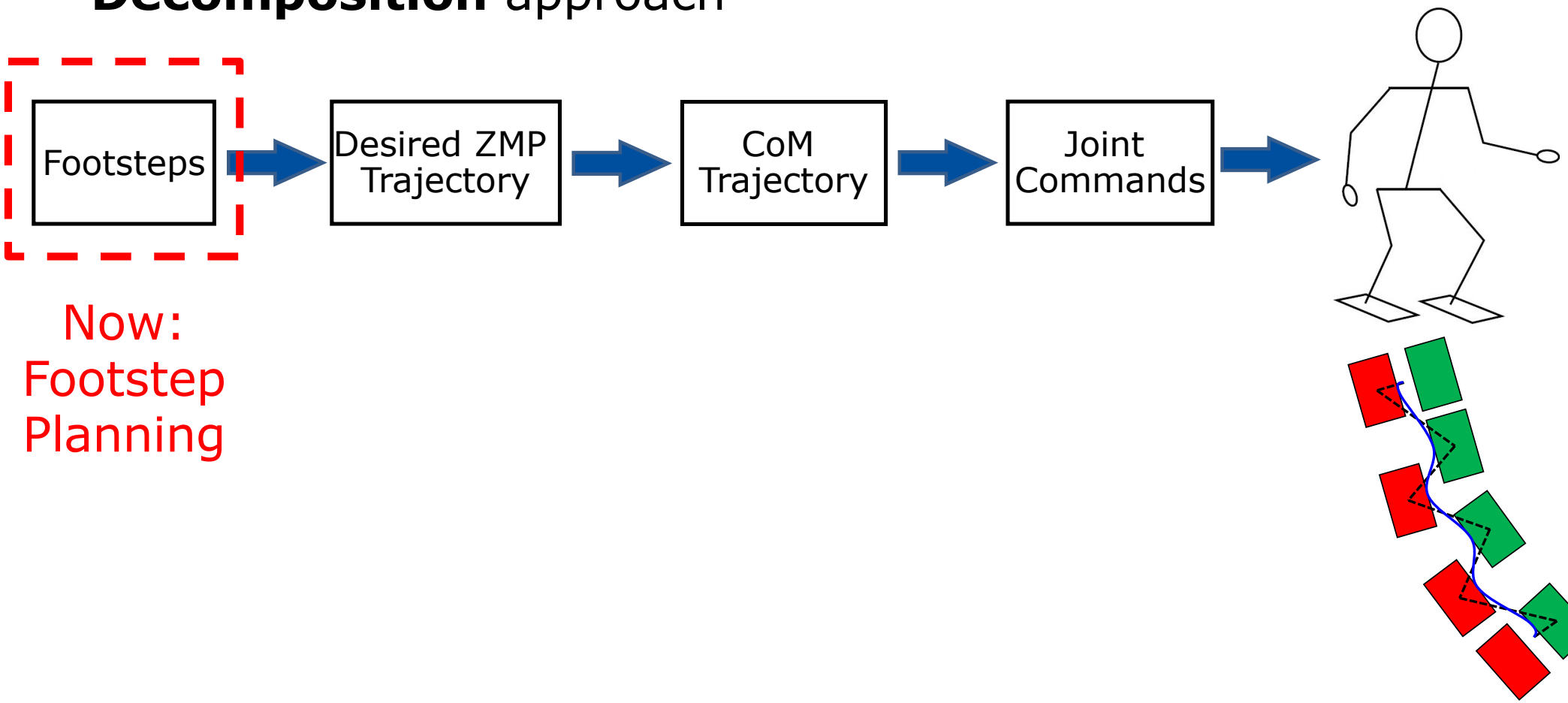
Recap – Motion Planning

- **Decomposition** approach



Recap – Motion Planning

- **Decomposition** approach



Footstep Planning on Flat Terrain: Search-Based Methods

A* Search Basics (1)

- Best-first search to find a cost-optimal path to the goal state on a graph
- Expands the states according to the **evaluation function**: $f(n)=g(n)+h(n)$
- $g(n)$: **actual costs** from start to state n
- Heuristics $h(n)$: **estimated costs** from state n to the goal
- **Expand a node:**
  - Generate its successors
  - Compute f-values of successors
  - Insert nodes into the priority queue, or update their f-value if they are already in the queue

A* Search Basics (2)

- Expand states, starting from the current robot state
- Sort successor states in priority queue according to *f*-value
- In each iteration, expand the state with the minimum *f*-value until the minimum is the desired goal state
- Each node keeps a pointer to its parent node to keep track of how it was found
- From that, the solution path can be reconstructed at the end

A* Search Basics (3)

- Optimal path = path with the minimum accumulated *g*-cost from the start to the goal state
- A* yields the **optimal path** if ***h* is admissible**
  (proof: see AI book of Russell/Norvig)
- Let $h^*(n)$ be the actual cost of the optimal path from n to the goal
- Heuristics h is **admissible** if the following holds for all n:
  $h(n) \leq h^*(n)$

A* Algorithm

```
OPEN = priority queue containing START
CLOSED = empty set
while lowest rank in OPEN is not the GOAL:
  current = remove lowest rank item from OPEN
  add current to CLOSED
  for neighbors of current:
    cost = g(current) + movementcost(current, neighbor)
    if neighbor in OPEN and cost less than g(neighbor):
      remove neighbor from OPEN, because new path is better
    if neighbor in CLOSED and cost less than g(neighbor):
      remove neighbor from CLOSED
    if neighbor not in OPEN and neighbor not in CLOSED:
      set g(neighbor) to cost
      add neighbor to OPEN
      set priority queue rank to g(neighbor) + h(neighbor)
      set neighbor's parent to current

reconstruct reverse path from goal to start
by following parent pointers
```

touched states

already expanded states

expand state

calculate f-value of successor

state touched before, check g-cost

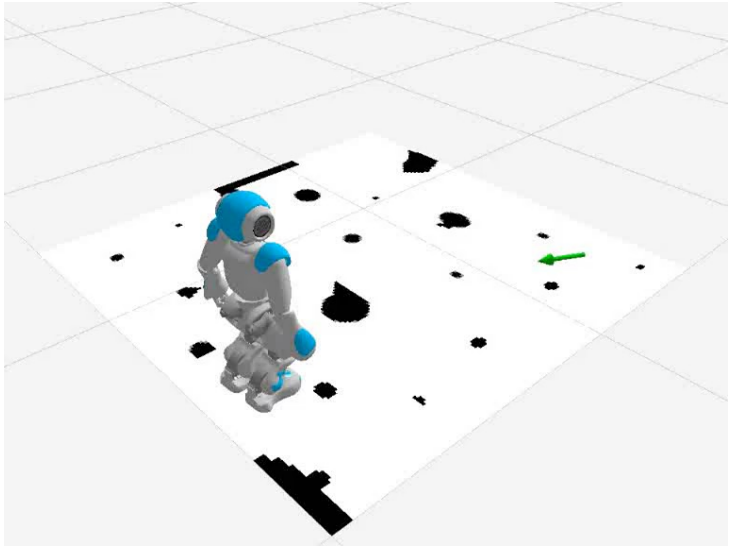
does not happen with admissible h

add state with its (new) cost

source: http://theory.stanford.edu/~amitp/GameProgramming/ImplementationNotes.html

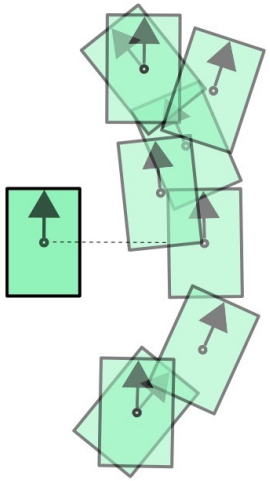
Footstep Planning with A*

- Search space: foot poses (x,y,θ)
- State: position and orientation of the stance foot
- Given: **discrete set of possible footsteps** wrt. stance foot
- Goal: Find the optimal footstep path with A* to the goal state



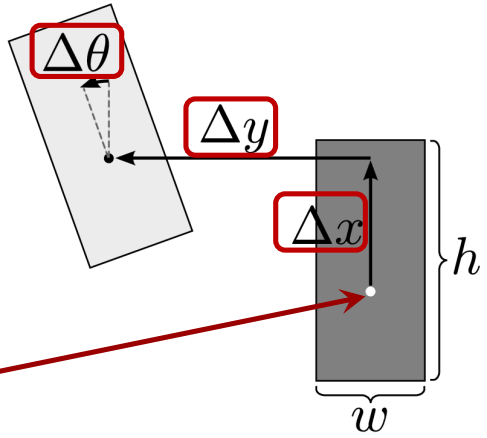
Footstep Planning with A*

- Fixed set of footstep covering the reachable region
- Construct a search tree of successor states, starting from the current state
- Check foot placements for collisions with obstacles during expansion



N. Perrin, "Biped footstep planning", Humanoid Robotics: A Reference, 2018

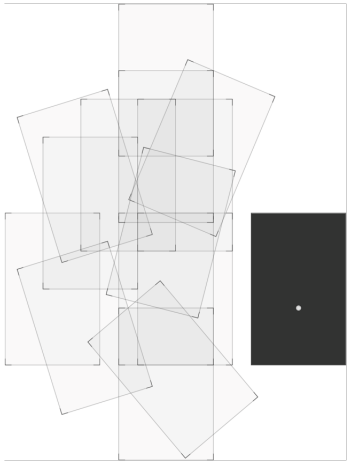
Footstep Actions & Costs



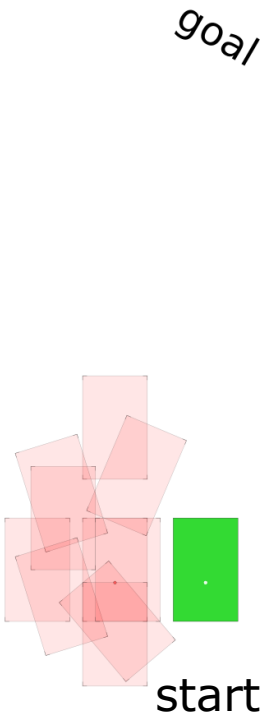
- State $s = (x, y, \theta)$
- Footstep action $a = (\Delta x, \Delta y, \Delta\theta)$
- Fixed set of footsteps $F = \{a_1, \ldots, a_n\}$
- Successor state $s' = t(s, a)$
- Transition costs:

$$c(s, s') = \|(\Delta x, \Delta y)^\top\|$$

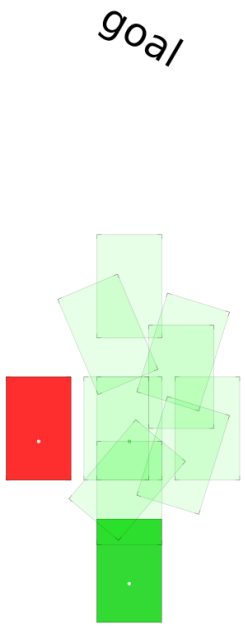
used to calculate g-value



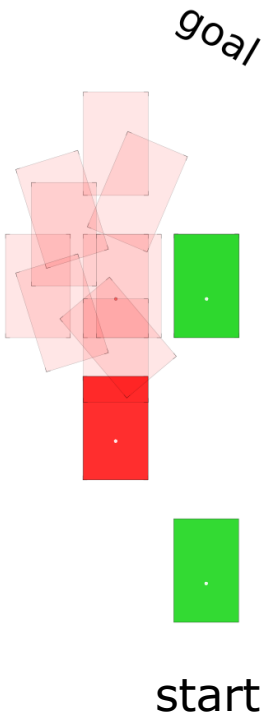
Footstep Planning



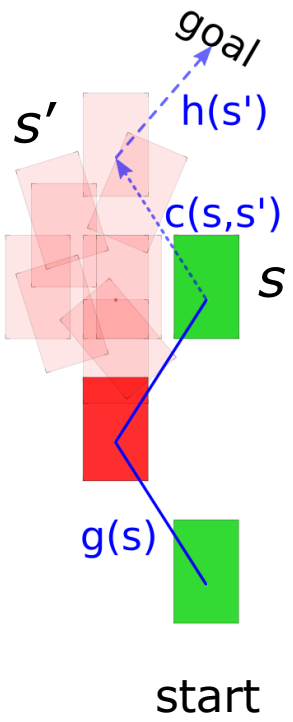
Footstep Planning



Footstep Planning



Footstep Planning



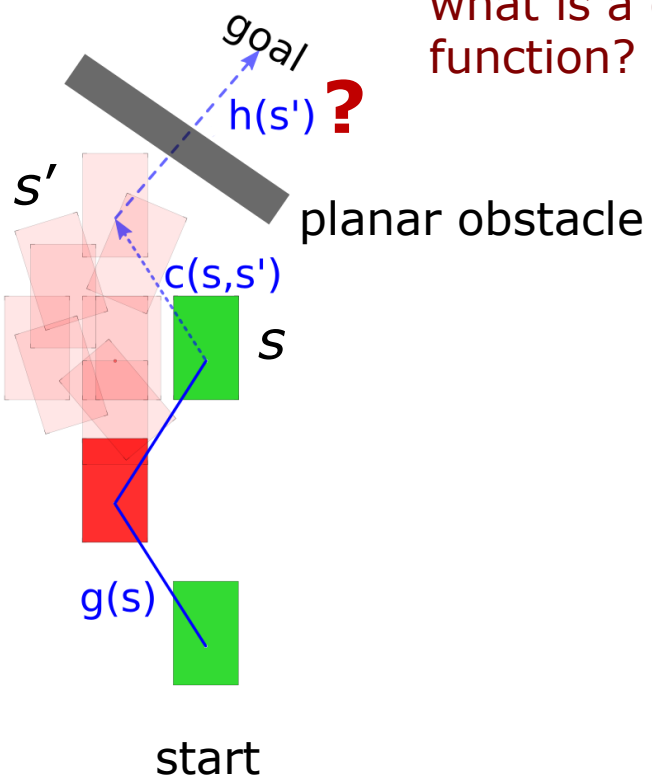
estimated costs from *s'* to goal

transition costs

path costs from start to *s*

Footstep Planning

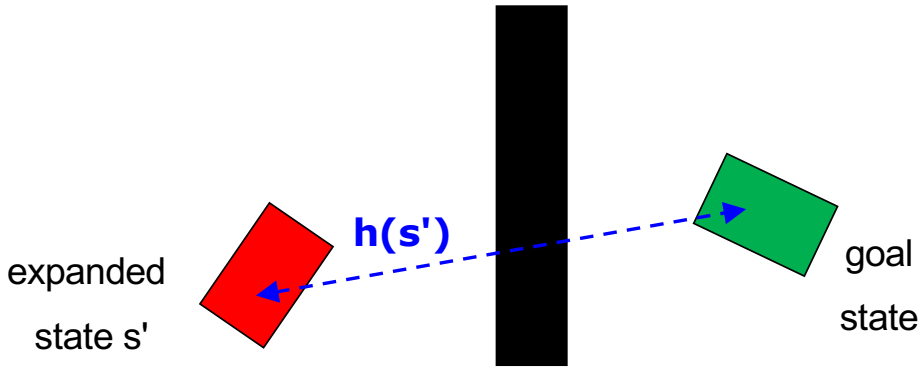
obstacles can create local minima in the search space



what is a good heuristic function?

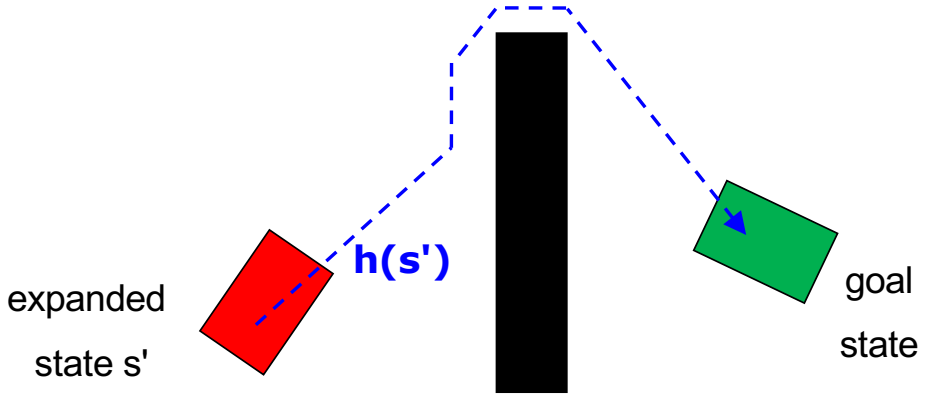
Heuristics for Footstep Planning

- **Heuristic highly influences the A* performance**
- Estimate the costs to the goal scaled with the **maximum forward step size**
- Typical heuristic functions are based on:
  1. Euclidean distance (straight line)



Heuristics for Footstep Planning

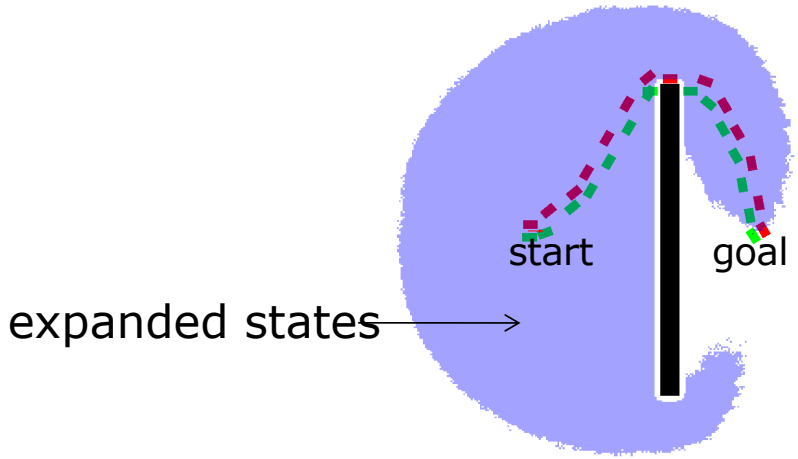
- **Heuristic highly influences the A* performance**
- Estimate the costs to the goal scaled with the **maximum forward step size**
- Typical heuristic functions are based on:
  1. Euclidean distance (straight line)
  2. Shortest 2D path with safety margin around obstacles



In general, shortest 2D path heuristic is inadmissible for humanoids… but yields good results in practice

Local Minima in the Search Space

- A* searches for the optimal path
- But sub-optimal results are often sufficient for navigation



With Euclidean distance
as heuristics

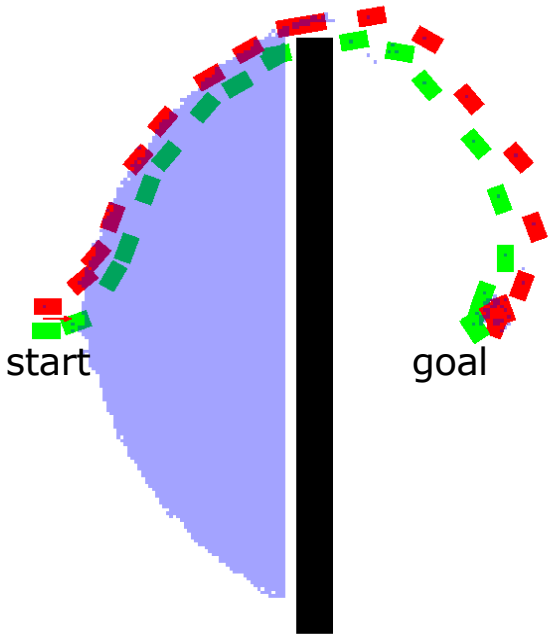
Anytime Repairing A* (ARA*)

- Weighted A* (wA*): **Heuristics "inflated" by a factor w**
- ARA* runs a **series of wA* searches**, iteratively lowering w as long as a given time limit is not met
- Resulting paths are **guaranteed** to cost no more than w times the costs of the optimal path
- ARA* reuses information from previous searches

Anytime Repairing A* (ARA*)

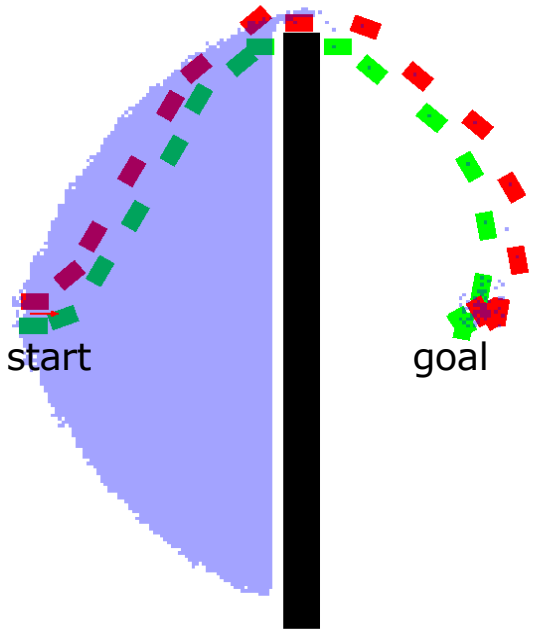
- An initially **large w** causes the search to find an initial non-optimal solution quickly
- Given enough time, ARA* finally searches with w = 1, producing an optimal path
- If the time limit is met before, the cost of the best solution found is guaranteed to be no worse than w times the optimal solution
- **ARA* finds a first sub-optimal solution faster than standard A* and approaches the optimal one as time allows**

ARA* with Euclidean Heuristic



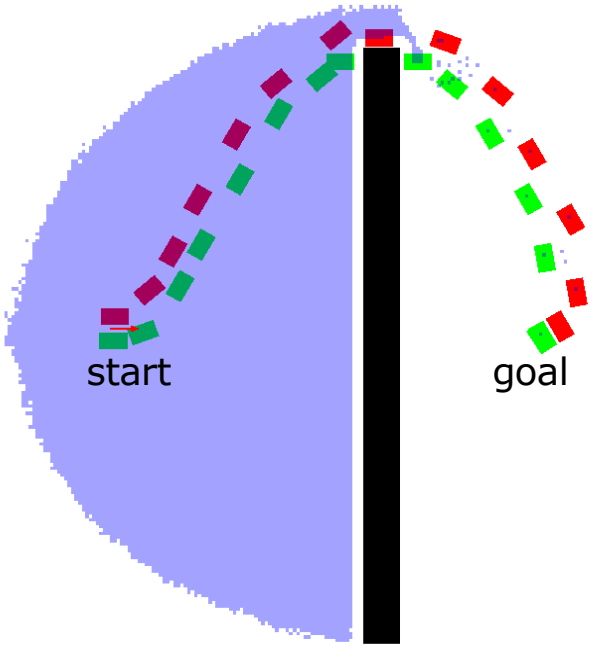
w = 10

ARA* with Euclidean Heuristic



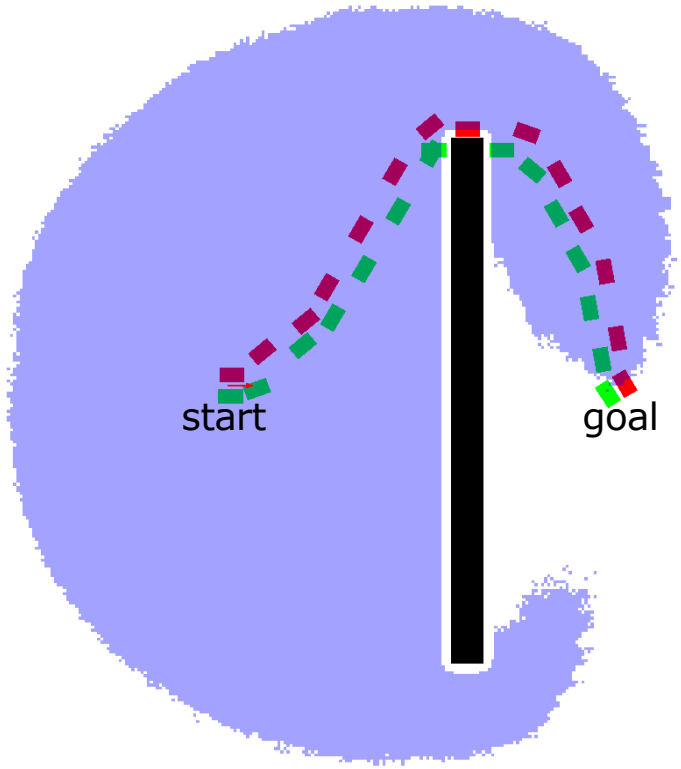
w = 5

ARA* with Euclidean Heuristic



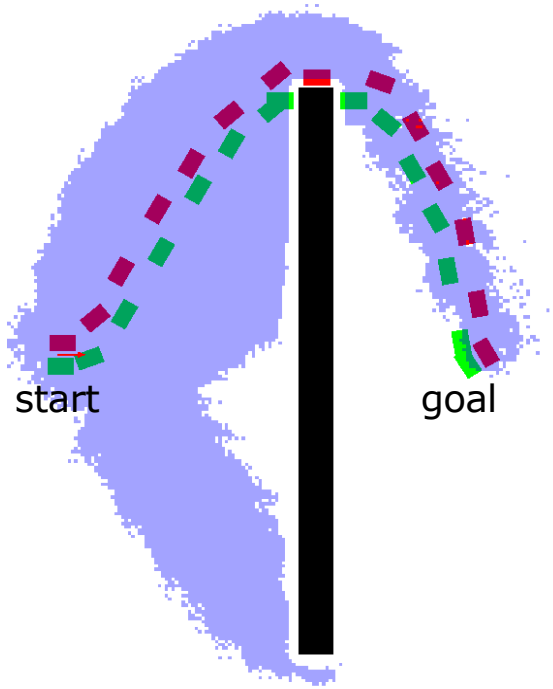
w = 2

ARA* with Euclidean Heuristic



w = 1

ARA* with Shortest 2D Path Heuristics



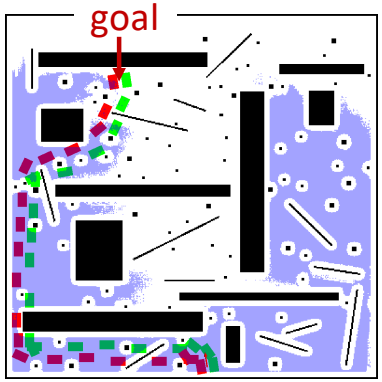
initial w=10

final w = 1

Planning in Dense Clutter Until a First Solution is Found

A*

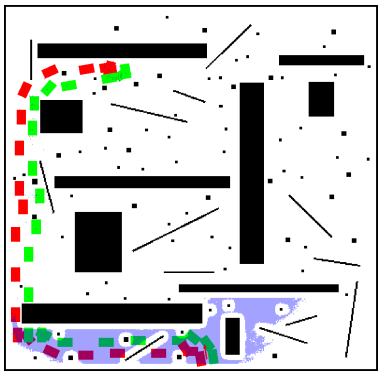
Euclidean heuristics



very high planning time

ARA* (w=5)

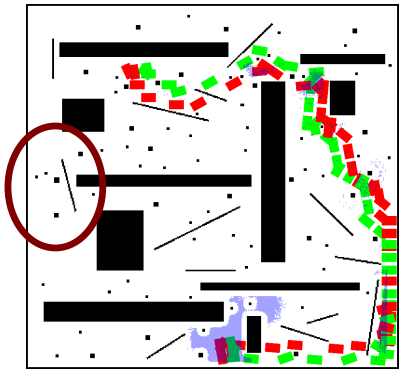
Euclidean heuristics



lower planning time

ARA* (w=5)

Shortest 2D path



very low planning time

2D path heuristics: inadmissible and may lead to longer paths, but expands much fewer states

Solution Quality and Search Time Depends on Heuristics

- Shortest 2D path heuristics: Can **overestimate costs** through narrow **cluttered regions**
- Shortest 2D path heuristics **leads to longer paths** but much **fewer expanded nodes** than Euclidean heuristics
- “Erosion” of oversteppable objects can be a solution (preprocessing of grid map)
- **Trade off** between **planning time** and **solution quality**

Footstep Planning on Uneven Terrain: Optimization Methods

Why Optimization?

- Handle **complex environments** with cluttered or uneven terrain
- **Heuristics are limited**, hard to scale and often fail in non-trivial settings
- Need for **constraints and models**, e.g., terrain geometry, robot kinematics, etc.
- **Generalization** to diverse terrain types, e.g., sloped, stairs, or rough terrains
- Principled decision-making under **physical and geometric constraints**

Fundamentals of Optimization

- Mathematical formulation

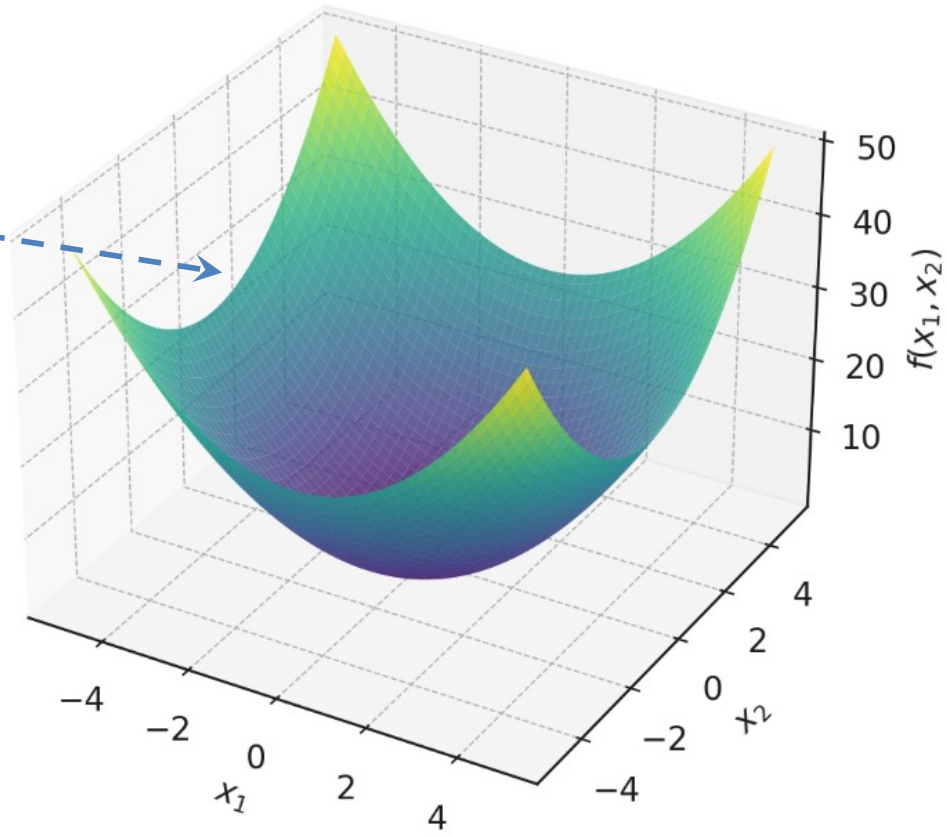
$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0. \end{aligned}$$

- $f(x)$: Cost (objective) function
- $g(x)$: Equality constraint
- $h(x)$: Inequality constraint

Fundamentals of Optimization

- Example

$$\begin{aligned} \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & x_2 - 1 - x_1^2 \geq 0, \\ & x_1 - 1 \geq 0. \end{aligned}$$



Fundamentals of Optimization

- Example

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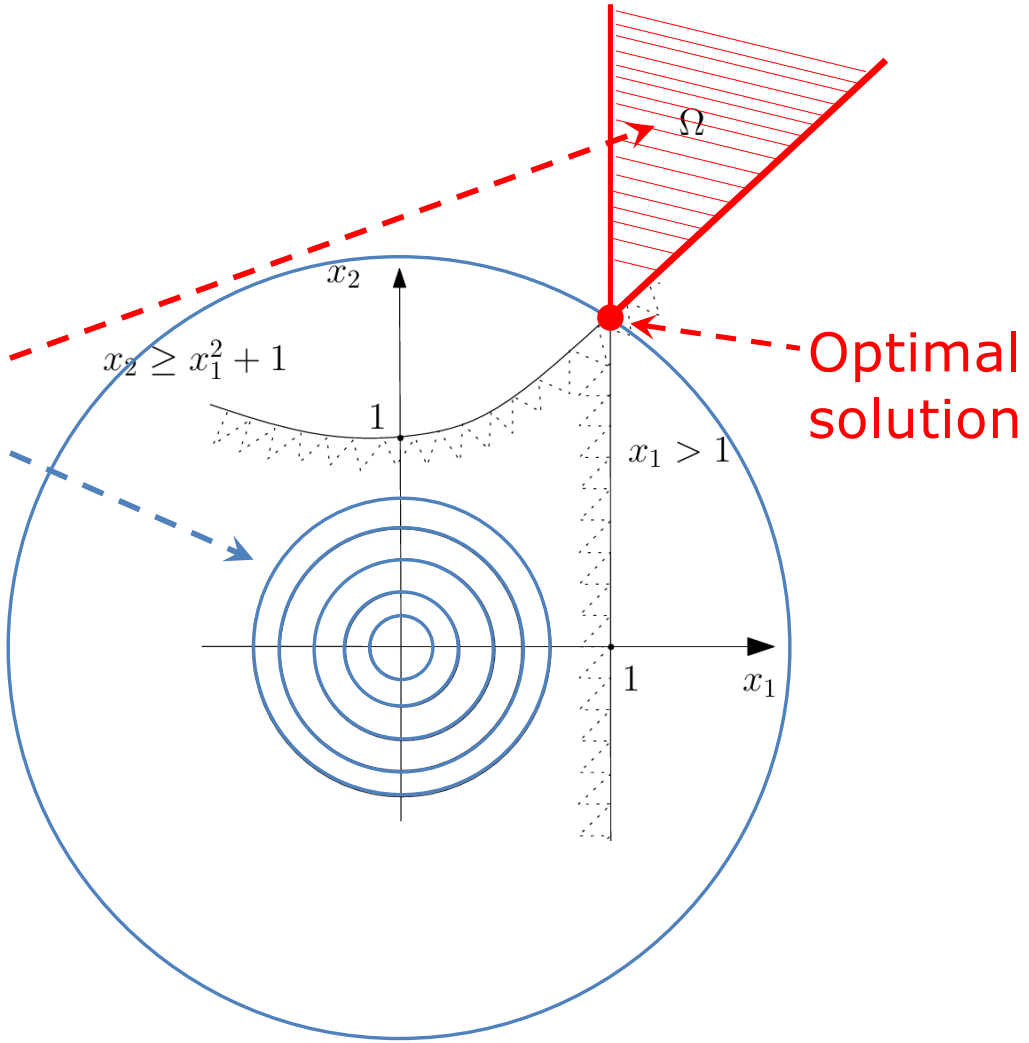


- Feasible solutions lie inside the region where constraints are satisfied (red dashed area)

Optimization Classes

- Linear Programming (LP)

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0. \end{aligned}$$

- **Linear cost function** and **constraints**

Optimization Classes

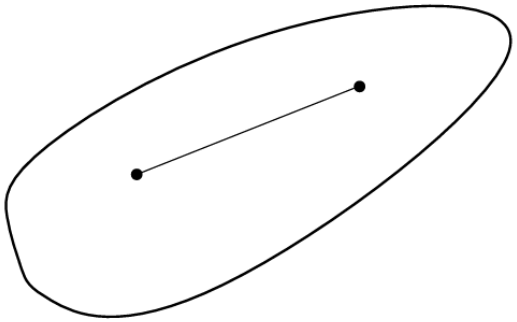
- Quadratic Programming (QP)

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x + \frac{1}{2} x^T B x \\ \text{subject to} \quad & Ax - b = 0, \\ & Cx - d \geq 0. \end{aligned}$$

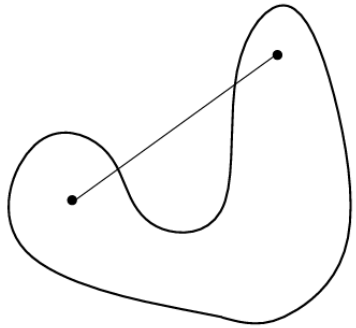
- **Quadratic cost function** and **linear constraints**

Convex Optimization

- **Convex set:** Any line connecting two points in the set lies inside the set



Convex

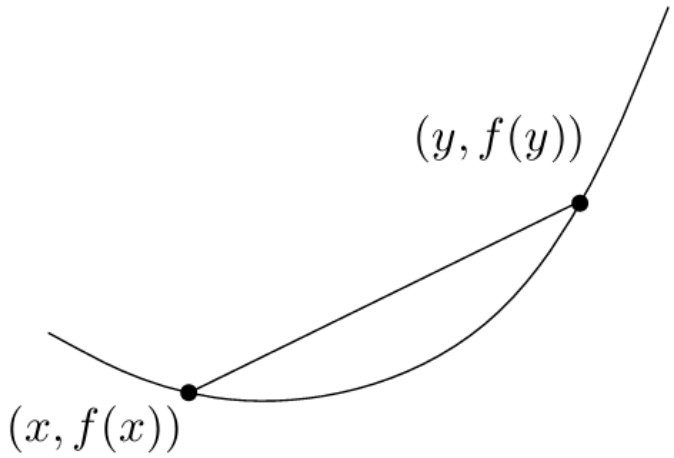


Nonconvex

Convex Optimization

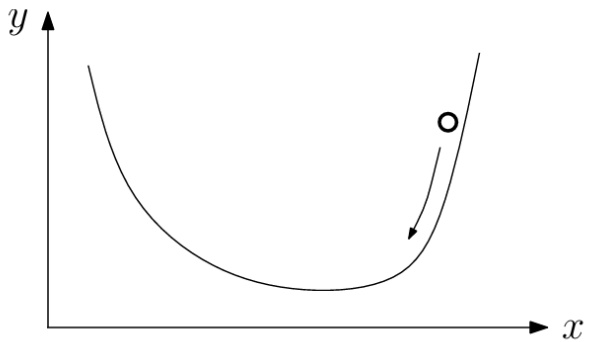
Convex function

- Function defined on a convex set
- Any line segment between two points on the graph of the function lies above or on the graph

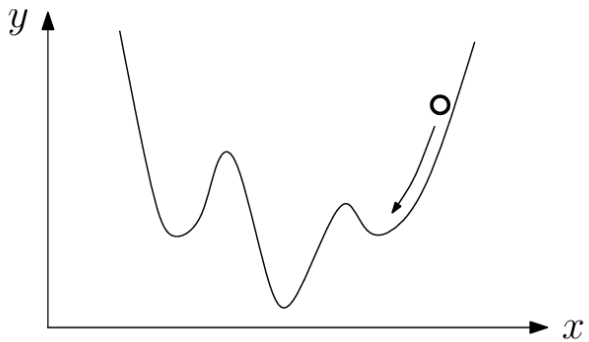


Convex Optimization

- Both cost function and constraints are convex
- Convexity important since in convex optimization, we have a **global minimum**



Convex



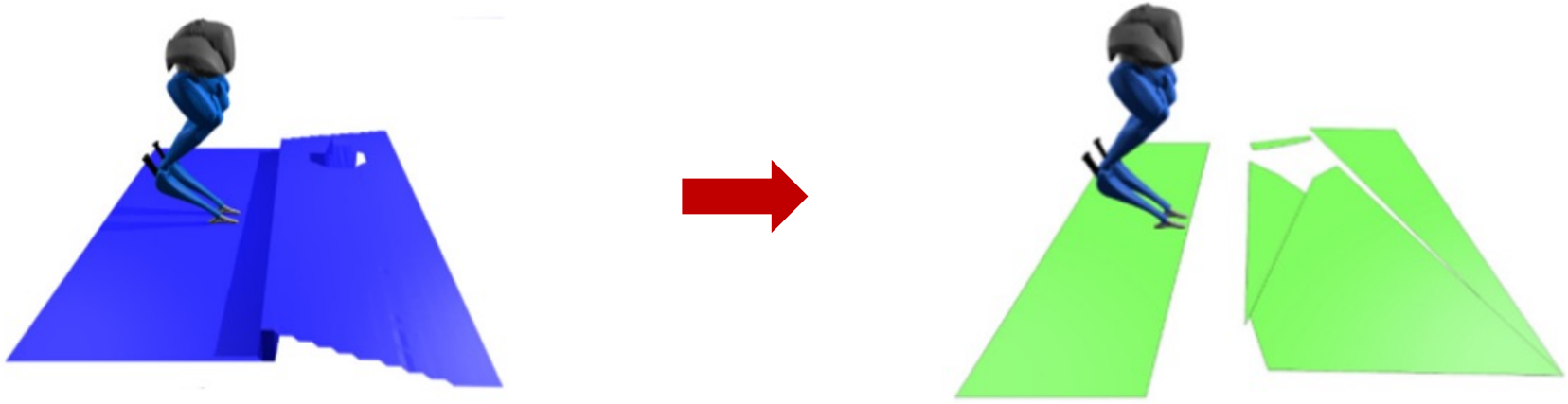
Non-convex

Footstep Planning on Uneven Terrain - Challenges

- **Imperfect** perception and state estimation
- Need to extract **safe regions** for online footstep planning from perception
- General terrain shape is **non-convex**
- Find a **segmentation** representing **steppable regions**
- Need to extract **convex hulls**, so that footstep planning can be formulated as convex optimization

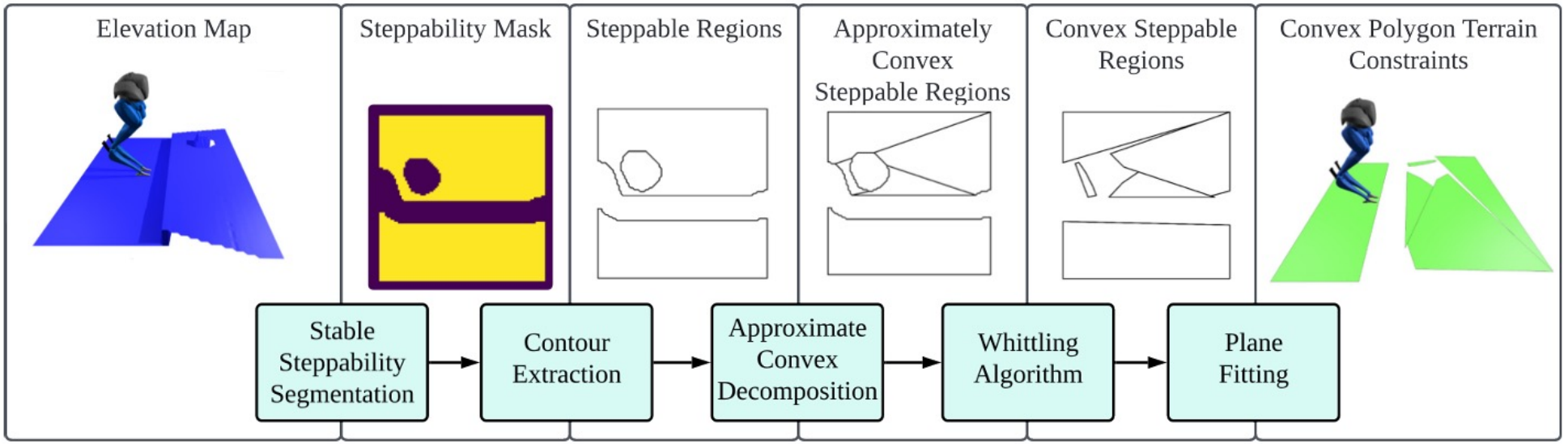
Perception on Rough Terrain

- Idea: Convert an elevation map to **convex polygons** representing terrain constraints, i.e., steppable areas



Acosta & Posa, "Perceptive mixed-integer footstep control for underactuated bipedal walking on rough terrain", arXiv, 2025

Perception on Rough Terrains



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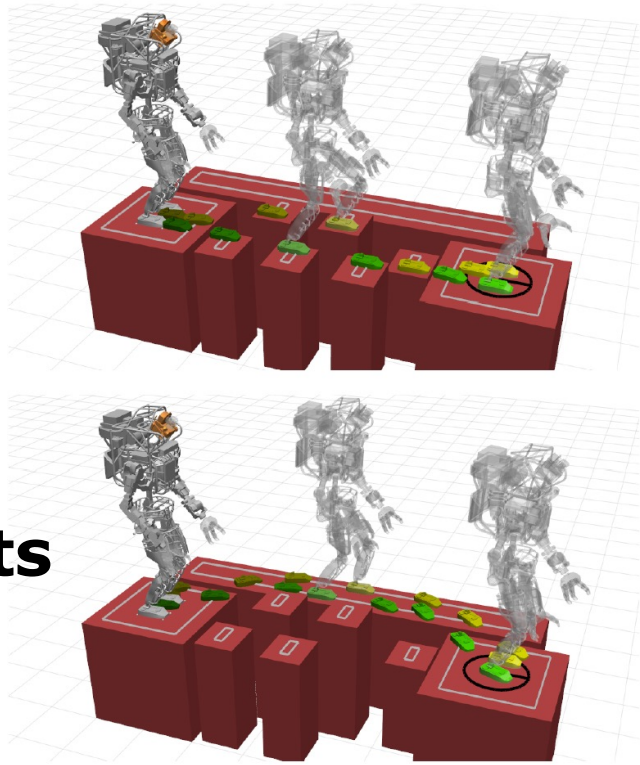
Footstep Planning with Convex Regions

- Mixed-Integer Convex Optimization (MICP)
- Terrain is preprocessed into convex safe regions
- How MICP works:
  1. Select **valid foot placements** inside preprocessed safe zones
  2. Plan toward a goal while respecting **reachability** and **terrain constraints**



Deits & Tedrake, "Footstep planning on uneven terrain with mixed-integer convex optimization", Humanoids, 2014

Footstep Planning with Convex Regions

- Convex region extraction: segment the terrain into **safe, flat convex regions**
- Step assignment via **integer** variables: each footstep is constrained to fall into **one of the convex regions**
- **Optimization problem**:
  - Minimize cost (e.g., distance to goal)
  - Considering constraints (e.g., footstep lies within assigned region, kinematic reachability between steps)

Footstep Planning with Convex Regions

- Determine the position $p = (x, y, z)$ and orientation θ of *N* footsteps, subject to the **constraints** so that:
  1. Each step does not intersect any obstacle
  2. Each step is within some convex reachable region relative to the pose of the prior step
- **Formulate** the problem as assigning each footstep to one of the precomputed **safe convex regions** of **obstacle-free** terrain

Footstep Planning with Convex Regions

- Simplified **objective function**:

$$\min_{p_i, \theta_i, z_{ij}} \sum_{i=1}^{N} \left(\|p_i - p_{i-1}\|^2 + c_{\text{turn}}(\theta_i, \theta_{i-1}) + c_{\text{goal}}(p_i) \right)$$

- Encourages short, smooth, goal-directed paths
- Turning cost **penalizes heading changes**
- Goal cost **encourages progress**
- z_{ij}: binary indicator if step i is in convex region j

Footstep Planning with Convex Regions

Constraints

1. Footstep in feasible convex region

$$A_j p_j \leq b_j \text{ only if } z_{ij} = 1$$

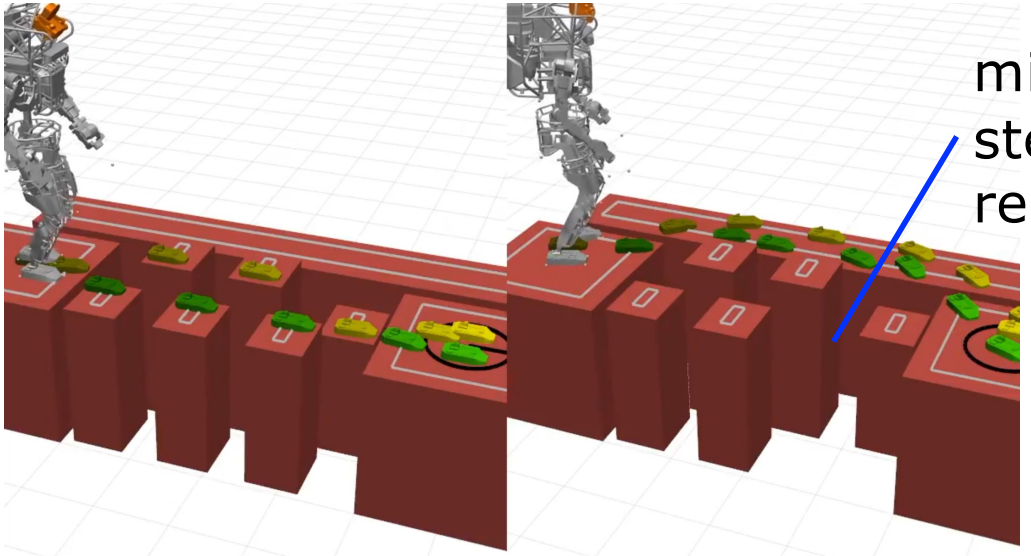
   - A_j, b_j : define polygon j from perception
   - Planner selects which region to step in via z_{ij}

2. Step reachability constraints
   - Enforces maximum step size and turning angle

$$\|p_i - p_{i-1}\|_2 \leq d_{\max}, \quad \angle(\theta_i - \theta_{i-1}) \leq \theta_{\max}$$

Footstep Planning with Convex Regions

- Handles **arbitrary terrain**
- Adaptable to changing convex regions
- Can incorporate different constraints
- But **trade-off** between computation time and generality



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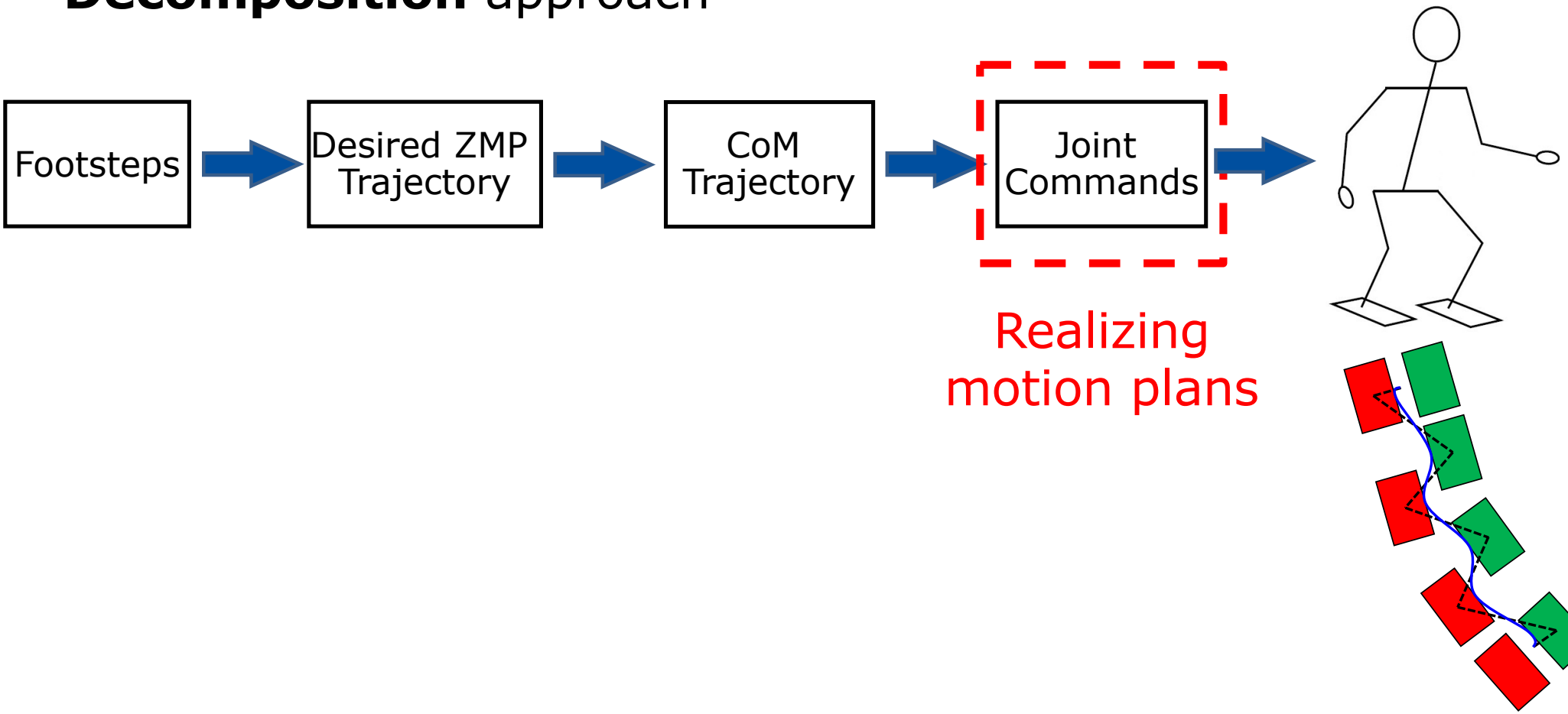
Realizing Motion Plans: Whole-Body Control

Recap – Motion Planning

- **Decomposition** approach



Whole-Body Control

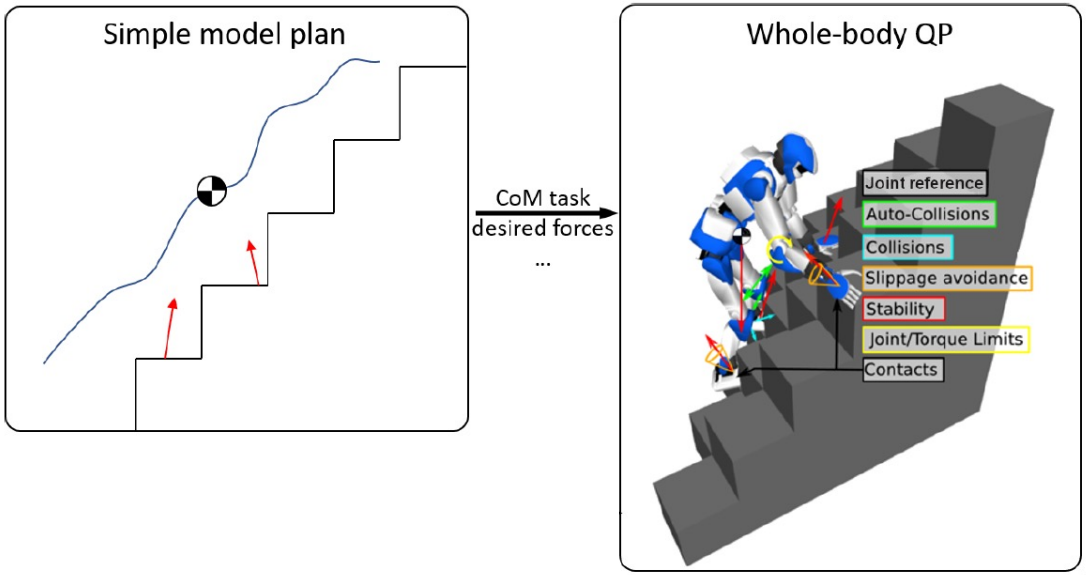
- Footstep planning and MPC generate desired footsteps and center of mass
- Complex optimization problems may be too slow for real-time control loops
- **Stabilizing controller** needed to execute and adapt the motion on the robot
- Whole-body control **enforces constraints ignored during planning** (e.g., joint limits, torque bounds, self-collisions, contact forces)

Whole-Body Control

- Formulated as **Hierarchical Quadratic Programming** (HQP)
- Use small convex QPs to compute joint torques or accelerations from state feedback
- Runs at high frequency (500–1,000 Hz) for real-time control

Whole-Body Control (WBC) using HQP

- Efficiently handles multiple constraints, e.g., COM trajectory, contacts, collisions, joint limits, torque bounds
- Enables **prioritization** of tasks (e.g., balance > foot tracking > posture)



Wensing et al.,"Optimization-Based Control for Dynamic Legged Robots", IEEE Transactions on Robotics, 2024

WBC Formulation

- Prioritized task space control as **optimization**
- Many control tasks (e.g., end-effector motion, contact forces, joint torques) can be written as **linear equations**
- These tasks are formulated as **least-squares problems**
- A general task takes the form: $A_i\, x = \ b_i$, where x is the control variable (e.g., joint accelerations or torques)
- Tasks are **stacked by priority levels**, with $i = 1$ being the highest priority

WBC Formulation

- Each task is solved as:

$$\min_{\boldsymbol{x}} \ \|A_i \boldsymbol{x} - \boldsymbol{b}_i\|^2$$

- While **preserving** the solutions of **higher-priority tasks**
- Leads to a **hierarchical optimization** approach
- Can be solved with:
  1. Iterative null-space projections (solution of each task has to be inside the null-space of the previous task)
  2. Sequential Quadratic Programs (QPs)

WBC Formulation

- Sequential quadratic programs
- Each control task is formulated as a least-squares objective
- Lower-priority tasks are solved **subject to equality constraints** from higher-priority tasks

$$\min_{\boldsymbol{x}} \|A_i \boldsymbol{x} - \boldsymbol{b}_i\|^2 \text{ subject to } A_1 \boldsymbol{x} = \boldsymbol{b}_1 \ \ldots A_{i-1}\, \boldsymbol{x} = \boldsymbol{b}_{i-1}$$

- Enables structured and interpretable control by **preserving task priorities**
- Can be solved efficiently using **standard QP solvers**

Minimal Example

- Control a **3-DoF planar** robotic arm with:
  1. **Task 1** (high priority): Move the end-effector to a target position
  2. **Task 2** (low priority): Control the joint angles for a desired posture
- Classic example of **operational space control**
- Control end-effector position in 2D (x, y) → 2 constraints
- Use the remaining 1 DoF for the secondary task (joint posture)
- Define $\boldsymbol{x}$ as the vector of joint accelerations $\ddot{\boldsymbol{q}}$

Minimal Example

Task 1 (end-effector position)

- **Goal**: Move the end-effector to a target position

$$\min_{\ddot{\boldsymbol{q}}} \left\| A_1 \ddot{\boldsymbol{q}} - \boldsymbol{b}_1 \right\|^2$$

- Where A_1 is a matrix that relates joint accelerations to the end-effector's position, and b_1 is the target position

Minimal Example

Task 2 (posture control)

- **Goal**: Achieve a desired posture (joint angle targets)

$$\min_{\ddot{\boldsymbol{q}}} \ \|A_2\ddot{\boldsymbol{q}} - \boldsymbol{b}_2\|^2$$

- Where A_2 is a matrix that relates joint accelerations to the joint positions, and b_2 is the desired posture

Minimal Example – Hierarchical QP Formulation

- Solve Task 2 (posture) while preserving Task 1 (end-effector)

$$\min_{\ddot{\boldsymbol{q}}} \ \|A_2\ddot{\boldsymbol{q}} - \boldsymbol{b}_2\|^2 \ \text{subject to} \ A_1\ddot{\boldsymbol{q}} = \boldsymbol{b}_1$$

- Here, Task 1 is enforced as a constraint to ensure that the **end-effector position is prioritized** over the posture

Minimal Example – Interpretation

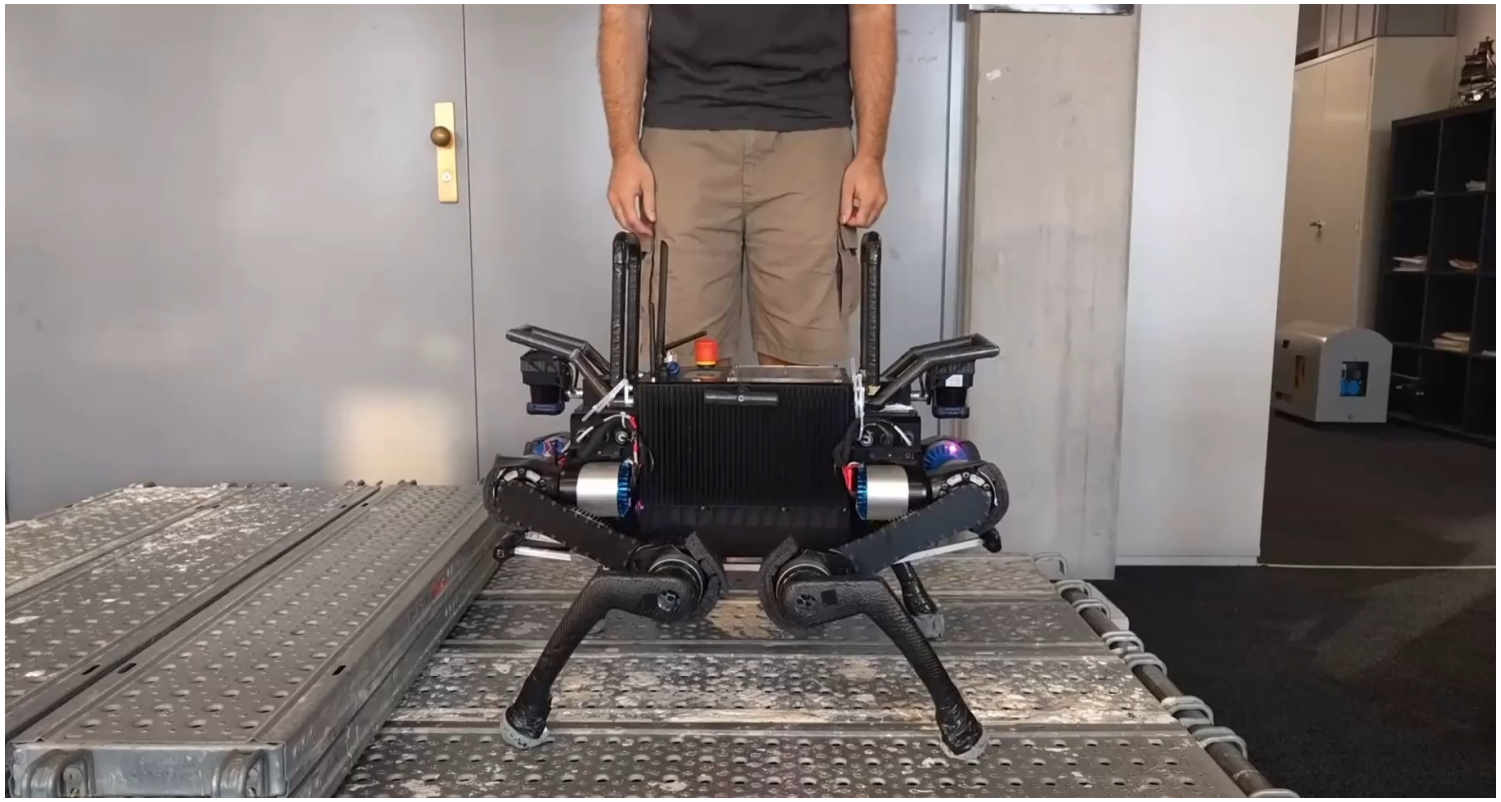
- The first task (end-effector position) has higher priority, so we enforce it as a **hard constraint**
- The second task (posture control) has lower priority, and we **minimize the error** for this task **while respecting the first task's constraint**

- This is a constrained optimization problem, and we can solve it using a **standard QP solver** to find the joint accelerations $\ddot{q}$ that satisfy both tasks

Implementation on Legged Robots

- **Complex locomotion behaviors** emerge from low-level control, by combining multiple inequality and equality tasks in a hierarchical optimization
- **Robot-specific constraints** (e.g., torque limits, joint limits, …) are directly encoded as inequality constraints
- **No explicit motion planning**: the controller continuously adapts to terrain and contact conditions through real-time task prioritization
- **Natural terrain adaptation** results from the structure of the control problem — not from high-level planning

Implementation on Legged Robots



Dario Bellicoso et al., “Perception-less terrain adaptation through whole body control and hierarchical optimization”, Humanoids, 2016

Summary

- Search-based footstep planning with **A* on flat terrain**
- **Heuristics** influence planning performance
- Footstep planning on uneven terrain with **Mixed-Integer Convex Optimization** (MICP)
- Terrain perception is used to extract **convex steppable regions** as constraints
- **Operational Space Control** (OSC) provides a framework to prioritize tasks in robot control
- **Whole-body control** coordinates all limbs and joints through **hierarchical optimization**, enabling stable locomotion and terrain adaptation

Evaluation

https://k.fachschaft.info/GyA9i

Literature

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